

# $\omega$ -Periodic graphs

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## Abstract

$\omega$ -periodic graphs are introduced and studied. These are graphs which arise as the limits of periodic extensions of the nearest neighbor graph on the integers. We observe that all bounded degree  $\omega$ -periodic graphs are amenable. We also provide examples of  $\omega$ -periodic graphs which have exponential volume growth, non-linear polynomial volume growth and intermediate volume growth.

## 1 Introduction

In [8] Milnor asked the following question. Does there exist a finitely generated group  $G$  such that the volume of the ball of radius  $n$  about the identity of Cayley graph of  $G$  grows faster than polynomially but slower than exponentially? This question was answered by Grigorchuk, who constructed a family of groups whose Cayley graphs have intermediate growth [5]. See [7] for a nice description of these groups.

Graphs that have intermediate volume growth also have a strong connection with long range percolation models in probability. In long range percolation on  $\mathbb{Z}$ , a random graph is constructed with  $\mathbb{Z}$  as its vertex set. The measure is determined by a sequence  $p_n$ . For each pair  $(u, v) \in \mathbb{Z} \times \mathbb{Z}$  there exists an edge  $e_{u,v}$  between  $u$  and  $v$  with probability  $p_{|u-v|}$ . The existence of an edge between  $u$  and  $v$  is determined only by the distance between  $u$  and  $v$  and is independent of the existence of edges between any other pairs of vertices. Long range percolation on  $\mathbb{Z}$  was introduced and studied in [10], [9] and [1] and is commonly used as a model for social networks.

Given a sequence  $p_n$  these papers studied the probability that an infinite connected subgraph exists. The more recent papers [3], [2], [6] and [4] considered the case when there is a unique infinite connected subgraph a.s. and studied the volume growth of this graph. In particular they considered the case that  $p_1 = 1$  and  $p_n = \beta n^\alpha$  for  $n > 1$ . For these sequences when  $\alpha > 2$  the random graph has linear volume growth a.s. When  $2 > \alpha > 1$ , the random graph has

intermediate growth a.s., yet large intervals admit polynomially small boundaries. And when  $\alpha \leq 1$ , all degrees are infinite. It is conjectured in [2] that when  $\alpha = 2$  one gets polynomial volume growth with powers depending on  $\beta$ .

In this paper we present a simple and natural graph,  $G$ , that has intermediate volume growth. Our graph is in some sense a hybrid of the Cayley graphs of the Grigorchuk groups and the graphs from long range percolation. Like the graphs generated by long range percolation our graph is constructed as an extension of the nearest neighbor graph on  $\mathbb{Z}$ . However its description is deterministic and, like Cayley graphs, has much more regularity than those random graphs. It does not have the full symmetry that Cayley graphs possess, it has small bottlenecks and in particular it is not transitive.

In addition to the study of one particular graph we also consider a broad family of graphs that contains  $G$ . This is the set of all graphs which are constructed as limits of periodic graphs on  $\mathbb{Z}$ . We call these  $\omega$ -periodic graphs. We study some of the coarse geometric properties shared by all  $\omega$ -periodic graphs and then give a few more examples that illustrate the possible volume growth of  $\omega$ -periodic graphs. In particular we show that there are  $\omega$ -periodic graphs of (non-linear) polynomial growth as well as ones with exponential volume growth.

## 2 The Basic Example

The vertices of  $G$  are the integers,  $\mathbb{Z}$ . Define the sets of edges

$$E_0 = \{(i, i + 1) : i \in \mathbb{Z}\}$$

and

$$E_k = \{(2^k(n - 1/2), 2^k(n + 1/2))\},$$

for all  $n \in \mathbb{Z}$  and  $k > 0$ .

The graph  $G$  has edges  $E = \cup_{k \geq 0} E_k$ .  $G$  is  $\omega$ -periodic because it is the union of  $G_K$  which has edges  $\cup_0^K E_k$ . We refer to the edges in  $E_k$  as the  $k$ th layer. The degree of every vertex of  $G$  (except for 0) is 4. More useful than our description is the picture below.

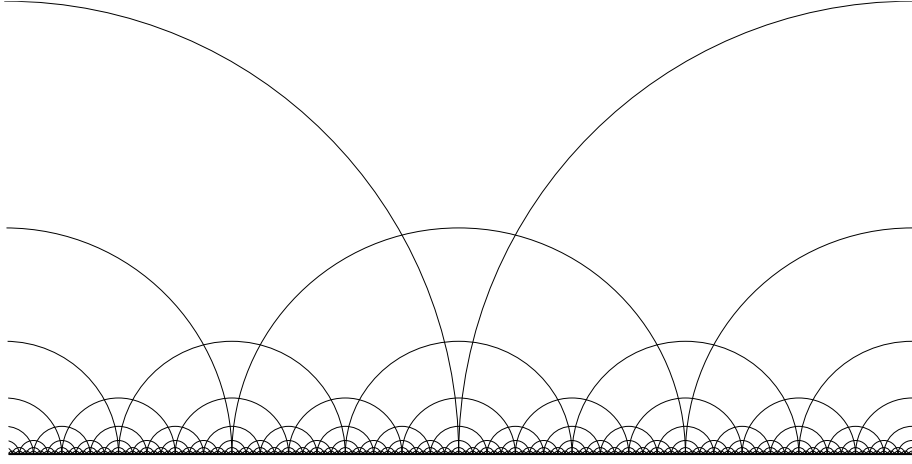


Figure 1: The basic example

In order to calculate the volume growth of  $G$  we would like to calculate the length minimal path between any two integers and then use that to estimate the volume growth. To find a minimal path between two integers  $u$  and  $v$  we take the following approach. We pick an integer  $k$  we move as quickly as possible from  $u$  to a vertex in the  $k$ th layer. Then we move in the  $k$ th layer and finally we go from the  $k$ th layer to  $v$ . It is easy to show that a minimal path must take such an approach.

The problem is given  $u$  and  $v$  how do we identify the optimal  $k$ . In general we do not know how to answer that question but we are able to calculate the length minimal paths between 0 and points of the form  $2^n$ . (This is called  $x_{n+1}$ .) Our main tools are induction and the symmetries of the graphs  $G_k$ . By knowing the distance from 0 to  $2^k$  for all  $k < n$  we can determine the distance from 0 to  $2^n$ . The inductive relationship is given in Lemma 1 while the formula is determined explicitly in Lemma 2.

Using this information along with the symmetries of  $G_k$  we are able to determine the growth rate of  $|B_j(0)|$ , the number of vertices within distance  $j$  of 0. Although  $|B_j(0)|$  does not have a simple formula we show in Lemma 4 that  $|B_j(0)| \approx j^{.5 \log j}$  and determine  $|B_j(0)|$  to within a factor of  $16j^2$ .

To analyze the growth rate of the ball centered at 0 we make the following definitions. For  $i \geq 1$  and  $j \in \mathbb{Z}$  let  $x_{i,j}$  be the distance from 0 to  $(j + 1/2)2^i$  in the graph  $G_i$ . The next lemma gives us an inductive relationship for  $x_i$  based on  $x_j$ ,  $j < i$ . In Lemma 2 we will calculate  $x_i$  explicitly.

**Lemma 1.**

$$x_{i,j} = x_{i,0} + |j + 1/2| - 1/2.$$

For all  $i > 2$

$$x_i = x_{i,0} = \min_{0 < k < i} 2x_k + 2^{i-k-1} - 1.$$

*Proof.* The proof is by induction on  $i$ . It is easy to check for  $i = 1, 2$  that the first formula is true. Fix  $i$  and  $j$ . Assume the lemma is true for all  $k < i$  and for all  $j$ .

Fix  $j$ . Let  $P$  be an oriented path in  $G_i$  from 0 to  $(j + 1/2)2^i$  which has minimal length. Let  $k$  be the largest integer such that an edge of  $P$  is in  $E_k$ . It causes no loss of generality to assume that  $k < i$ . (This is because there is a first point of the form  $(j' + 1/2)2^i$  in  $P$ . If the lemma is not true for  $i$  and  $j$  then it is also not true for  $i$  and  $j'$ .)

Since  $P$  has minimal length and  $i > 2$  then it is clear that  $k > 0$ . Divide  $P$  up into three parts,  $P_1, P_2$ , and  $P_3$ , as follows. Let  $n_1$  be the first point in  $P$  of the form  $(n_1 + 1/2)2^k$ . Let  $n_2$  be the last point in  $P$  of the form  $(n_2 + 1/2)2^k$ . Then  $P_1$  is the portion of  $P$  connecting 0 to  $(n_1 + 1/2)2^k$ ,  $P_2$  connects  $(n_1 + 1/2)2^k$  to  $(n_2 + 1/2)2^k$ , and  $P_3$  connects  $(n_2 + 1/2)2^k$  to  $(j + 1/2)2^i$ . Then we have that

$$\begin{aligned} |P| &= |P_1| + |P_2| + |P_3| \\ &\geq x_{k,n_1} + (n_2 - n_1) + x_{k,2^{i-k-1}-n_2-1+j2^{i-k}} \\ &\geq x_k + |n_1 - 1/2| + 1/2 + (n_2 - n_1) + x_k + |2^{i-k-1} - n_2 + j2^{i-k} - 1/2| - 1/2 \\ &\geq 2x_k + n_1 + (n_2 - n_1) + 2^{i-k-1} - n_2 - 1 + j2^{i-k} \\ &\geq 2x_k + 2^{i-k-1} - 1 + j2^{i-k} \\ &\geq \min_{0 < k < i} 2x_k + 2^{i-k-1} - 1 + j2^{i-k}. \end{aligned}$$

The existence of a path of the minimum distance is easy to construct. □

We now calculate  $x_i$  exactly. Let

$$y_n = \frac{n^2 + 3n + 2}{2}$$

and

$$z_n = n2^n + 1.$$

**Lemma 2.** For all  $i$  and  $n$ , if  $y_n < i \leq y_{n+1}$  then

$$x_i = z_{n+1} - (y_{n+1} - i)2^n.$$

In particular  $x_{y_n} = z_n$ .

*Proof.* The proof is by induction. It is easy to check that the lemma is true for all  $i \leq y_1 = 3$ .

Now assume that the lemma is true for all  $j < i$ . Note that this implies that the sequence

$$x_j - x_{j-1}$$

is nondecreasing for all  $j$ ,  $2 \leq j \leq i - 1$ .

Let

$$f(k) = f(i, k) = 2x_k + 2^{i-k-1} - 1.$$

By Lemma 1  $x_i = \min_{k < i} f(k)$ . Let  $n$  be the largest integer such that  $y_n < i$ . We break the proof up into two cases. The first is when  $i < y_{n+1}$  and the second is when  $i = y_{n+1}$ .

**Case 1:** We show that the minimum of  $f(k)$  occurs at two values,  $i - n - 1$  and  $i - n - 2$ . More specifically we show that  $f(k)$  is decreasing up to  $i - n - 1$  and increasing afterwards. Let  $m = i - n - 1$ . Since  $y_n < i < y_{n+1}$  we have that  $y_{n-1} < m \leq y_n$ . Thus

$$x_m - x_{m-1} = 2^{n-1}$$

and

$$x_{m+1} - x_m \geq 2^{n-1}.$$

We now calculate for  $j < m$

$$\begin{aligned} f(j) - f(j-1) &= 2(x_j - x_{j-1}) + (2^{i-j-1} - 2^{i-j}) \\ &\leq 2(x_m - x_{m-1}) - 2^{i-j-1} \\ &\leq 2 \cdot 2^{n-1} - 2^{i-m} \\ &\leq 2^n - 2^{n+1} \\ &< 0. \end{aligned}$$

We also have that

$$\begin{aligned} f(m) - f(m-1) &= 2(x_m - x_{m-1}) + (2^n - 2^{n+1}) \\ &= 2 \cdot 2^{n-1} - 2^n \\ &= 0 \end{aligned}$$

For  $l > m$  we have

$$\begin{aligned}
f(l+1) - f(l) &= 2(x_{l+1} - x_l) + (2^{i-l-2} - 2^{i-l-1}) \\
&\geq 2(x_{m+1} - x_m) - 2^{i-l-2} \\
&\geq 2(2^{n-1}) - 2^{i-m-2} \\
&\geq 2^n - 2^{n-1} \\
&> 0.
\end{aligned}$$

From these three calculations it is clear that  $f$  obtains its minimum at  $m-1$  and  $m$ . It is easy to check that the induction hypothesis gives the right value for  $x_i$ .

**Case 2:** Now we have that  $i = y_{n+1}$ . We claim that in this case the unique minimum of  $f$  occurs at  $m = y_n$ . By the induction hypothesis we have that

$$x_m - x_{m-1} = 2^{n-1}$$

and

$$x_{m+1} - x_m = 2^n.$$

We now calculate for  $j < m$

$$\begin{aligned}
f(j) - f(j-1) &= 2(x_j - x_{j-1}) + (2^{i-j-1} - 2^{i-j}) \\
&\leq 2(x_m - x_{m+1}) - 2^{i-j-1} \\
&= 2 \cdot 2^{n-1} - 2^{i-m-1} \\
&= 2^n - 2^{n+1} \\
&= 2^n > 0.
\end{aligned}$$

For  $l > m$

$$\begin{aligned}
f(l+1) - f(l) &= 2(x_{l+1} - x_l) + 2^{i-l-2} - 2^{i-l-1} \\
&\geq 2(x_{m+1} - x_m) - 2^{i-l-2} \\
&\geq 2^{n+1} - 2^{i-m-2} \\
&\geq 2^{n+1} - 2^n \\
&= 2^n \\
&> 0.
\end{aligned}$$

From these calculations it is clear that  $f$  obtains its minimum at  $m$ .

Thus

$$\begin{aligned}
x_{y_{n+1}} &= 2z_n + 2^{n+1} - 1 \\
&= 2(n2^n) + 2^{n+1} - 1 \\
&= n2^{n+1} + 2^{n+1} - 1 \\
&= (n+1)2^{n+1} - 1 \\
&= z_{n+1}.
\end{aligned}$$

Thus the induction hypothesis is true for  $i$  and the lemma is proven.  $\square$

Now we use this information to estimate the growth rate of the ball around 0.

**Lemma 3.** *For any  $i > 0$  and any  $m$ ,  $0 \leq m \leq 2^{i-1}$  we have that  $d(0, m) \leq x_i$ .*

*Proof.* By induction we can see that the distance from any point to the nearest vertex of level  $E_k$  is at most  $x_k$ . For any  $m$  such that  $0 \leq m \leq 2^{i-1}$  the nearest vertex to  $m$  of level  $k$  will lie in the interval  $(0, 2^{i-1})$ . Thus

$$d(0, m) \leq \min_k 2x_k + 2^{i-k-1} - 1 = x_i.$$

$\square$

**Lemma 4.** *There is a function  $G(j)$  defined below ( $G(j) \approx j^{.5 \log j}$ ) such that*

$$G(j) \leq |B_j(0)| \leq 16j^2 G(j).$$

*Proof.* First for  $i > 2$  let

$$w_i = \sup_{k \geq 0} 2^{k-1} + 2^k((x_i - 1)/2 - x_k).$$

(We want  $i > 2$  because all  $x_i$  are odd except  $x_2 = 2$ .) Thus  $w_i$  is the largest integer such that there exists a path from 0 to  $w_i$  of length  $(x_i - 1)/2$ . This makes it clear that

$$B_{(x_i-1)/2}(0) \subset (-w_i, w_i).$$

Let  $P$  be a path of length  $(x_i - 1)/2$  connecting 0 to  $w_i$  and  $k$  be such that the longest step in  $P$  is of size  $2^k$ . Suppose that  $w_i \geq 2^{i-2}$ . The graph  $G_k$  is symmetric about any point of the form  $w_i \pm l2^{k-1}$ . Thus by combining  $P$  and the reflection of  $P$  (about some suitably chosen point) we could construct a path from 0 to  $2^{i-1}$  of length at most  $x_i - 1$ . This is a contradiction. Thus

$$w_i < 2^{i-2}$$

and

$$B_{(x_i-1)/2}(0) \subset (-2^{i-2}, 2^{i-2}). \quad (1)$$

On the other hand by Lemma 3 gives us that

$$[-2^{i-1}, 2^{i-1}] \subset B_{x_i}(0). \quad (2)$$

Plugging  $i = y_{n+1}$  into line (1) and  $i = y_n$  into line (2) gives

$$[-2^{y_n-1}, 2^{y_n-1}] \subset B_{x_{y_n}}(0) \subset B_{(x_{y_{n+1}}-1)/2}(0) \subset (-2^{y_{n+1}-2}, 2^{y_{n+1}-2}). \quad (3)$$

If  $x_{y_n} \leq j < x_{y_{n+1}}$  then

$$B_{x_{y_n}}(0) \subset B_j(0) \subset B_{y_{n+1}}(0)$$

and

$$[-2^{y_n-1}, 2^{y_n-1}] \subset B_j \subset (-2^{y_{n+2}-2}, 2^{y_{n+2}-2}). \quad (4)$$

We now rewrite this equation using the following definitions. Let  $f(j) = \sup_n z_n \leq j$ ,  $g(j) = 2^{f(j)-1}$ ,  $p(n) = (n^2 + 3n + 2)/2$ , and  $h(j) = 2^{p(f(j)+2)-2}$ . Thus line (4) becomes

$$[-g(j), g(j)] \subset B_j \subset (-h(j), h(j)). \quad (5)$$

Then calculating

$$\begin{aligned} \frac{h(j)}{g(j)} &= 2^{p(f(j)+2)-2+p(f(j))-1} \\ &= 2^{.5((f(j)^2+7f(j)+12)-(f(j)^2+3f(j)+2))-1} \\ &= 2^{2f(j)+4}. \end{aligned}$$

By the definition of  $z_n$  we get the bound

$$f(j)2^{2(j)} \leq j$$

and thus  $f(j) < \log(j)$ . Putting these two together we get that

$$\frac{h(j)}{g(j)} = 2^{2f(j)+4} < 2^{2\log(j)+4} = 16j^2.$$

Thus

$$[-g(j), g(j)] \subset B_j \subset (-16j^2g(j)j^2, 16j^2g(j)).$$

Thus we can pick  $G(j) = 2g(j) + 1$ . Finally we check that

$$G(j) \approx g(j) \approx 2^{.5f(j)^2} = j^{.5\log j}.$$

□



### 3 $\omega$ -periodic graphs

The graphs that we will consider are all obtain as the limit of periodic graphs.

**Definition 1.** A graph  $G$  with vertices labelled by  $\mathbb{Z}$  is  **$\omega$ -periodic** if it is a union of periodic graphs over  $\mathbb{Z}$ .

Our general result about the growth of  $\omega$ -periodic graphs is the following.

**Proposition 2.** Bounded degree  $\omega$ -periodic graphs are amenable.

*Proof.* Let  $G$  be an  $\omega$ -periodic graph with a uniformly bounded degree. To show amenability it is enough to present a growing sets for which the ration between the size of the boundary of the sets and the size of the sets is approaching 0.  $G$  is composed of periodic layers ordered according to the density of the vertices used. In particular the density of vertices that are connected more than  $k$  away, denoted by  $k(s)$  exists and is going to 0 with  $k$ . Hence if we consider a large interval of size  $n$ , we get that it's boundary is smaller than  $2k + k(s)n$ . Thus the ratio of the boundary to the interval can be made arbitrarily small.  $\square$

### 4 Polynomial Growth

In this section we will use a subgraph  $\tilde{G}$  of  $G$  in Section 2. Again the vertices of  $\tilde{G}$  are the integers,  $\mathbb{Z}$ , and we define the sets of edges

$$E_0 = \{(i, i + 1) : i \in \mathbb{Z}\}$$

and

$$E_k = \left\{ \left( 2^k(n - 1/2), 2^k(n + 1/2) \right) \right\},$$

for all  $n \in \mathbb{Z}$  and  $k > 0$ . We define the graph  $\tilde{G}$  to have edges  $E = E_0 \cup (\cup_{k \geq 0} E_{2^k})$ .

The proof that the volume of  $\tilde{B}_m(0)$  grows polynomially in  $m$  is almost exactly like the proof of the volume growth of the full graph in Section 2. First we calculate the distance  $\tilde{x}_{2^i}$  from 0 to  $2^{2^i-1}$ . Then we use this information to bound the volume growth. The difference is that we get the formula

$$\tilde{x}_{2^i} = \min_{0 < k < i} 2\tilde{x}_{2^k} + 2^{2^i-2^k-1} - 1.$$

We use the notation  $\tilde{B}_j(0)$  to be the ball of radius  $j$  in  $\tilde{G}$  and

$$\tilde{w}_{2^i} = \max_k 2^{2^k-1} + 2^{2^k}((\tilde{x}_{2^i} - 1)/2 - \tilde{x}_{2^k}).$$

This gives us the following lemma.

**Lemma 5.** 1.  $\tilde{x}_{2^i} = 2\tilde{x}_{2^{i-1}} + 2^{2^{i-1}-1} - 1$

2.  $2^{2^{i-1}-1} \leq \tilde{x}_{2^i} \leq 2^{2^i-1}$

3.  $[-2^{2^i-1}, 2^{2^i-1}] \subset \tilde{B}_{\tilde{x}_{2^i}}(0),$

4.  $B_{2\tilde{x}_{2^i}} \subset (-\tilde{w}_{2^i}, \tilde{w}_{2^i})$  and

5.  $\tilde{w}_{2^i} \leq 2^{2^i}.$

*Proof.* The proof of these facts goes exactly as the proof of the corresponding statements in Section 2.  $\square$

**Lemma 6.** If  $j = \tilde{x}_{2^i}$  then

$$j^2 \leq |\tilde{B}_j(0)| \leq 8j^2.$$

*Proof.* The lower bound follows from condition 3 and the lower bound in condition 2 in the previous lemma. The upper bound follows from conditions 4, 5 and the upper bound in condition 2.  $\square$

## 5 Exponential Growth

In this section we will construct an  $\omega$ -periodic graph that contains a dyadic tree. Thus the graph has exponential volume growth. Again we let

$$E_0 = \{(i, i+1) : i \in \mathbb{Z}\}$$

be the graph between adjacent integers.

Let  $p_i$  be the  $i$ th prime,

$$l_m = \prod_{i=1}^{i=2^m} (p_i)^i$$

and

$$t_{m,j} = (p_m)^j, \quad j = 1 \dots 2^{m-1}.$$

Notice that if  $t_{m,j} = t_{m',j'}$  then  $m = m'$  and  $j = j'$ .

Define the  $m$ th level by

$$E_m = \cup_{k \in \mathbb{Z}} \left( \cup_{j=1}^{2^{m-1}} ((t_{m,j} + kl_{m+1}, t_{m+1,2j-1} + kl_{m+1}) \cup (t_{m,j} + kl_{m+1}, t_{m+1,2j} + kl_{m+1})) \right).$$

Another way to describe  $E_m$  is as follows. Let

$$V_m = \cup_{j=1}^{2^{m-1}} t_{m,j}.$$

Also let  $L_m = V_m + \mathbb{Z}l_{m+1}$  and  $R_m = V_{m+1} + \mathbb{Z}l_{m+1}$ . Then every edge in  $E_m$  has its leftmost endpoint in  $L_m$  and its rightmost endpoint in  $R_m$ . Also every point in  $L_m$  is the left hand end point of two edges in  $E_m$ . Every point in  $R_m$  is the right hand end point of one edge in  $E_m$ .

First we show that the graph contains a dyadic tree and then we show that it has bounded degree.

**Lemma 7.** *There is a dyadic tree rooted at  $t_{1,1} = 2$ .*

*Proof.* Note that  $V_{m+1} \subset L_{m+1} \cap R_m$ . Then the  $2^m$  vertices at distance  $m$  from the root are  $V_{m+1}$ . □

**Lemma 8.**  $L_m \cap \left(\bigcup_{j=1}^{m-1} L_j\right) = \emptyset$  and  $R_m \cap \left(\bigcup_{j=1}^{m-1} R_j\right) = \emptyset$ .

*Proof.* Fix an  $m$ . Every element of  $L_m \bmod l_{m+1}$  is only divisible by powers of  $p_m$ . Every element of  $\bigcup_{j=1}^{m-1} L_j \bmod l_{m+1}$  is divisible by at least one prime less than or equal to  $p_{m-1}$ . Thus the two sets are disjoint.

Fix an  $m$ . Every element of  $R_m \bmod l_{m+1}$  is only divisible by powers of  $p_{m+1}$ . Every element of  $\bigcup_{j=1}^{m-1} R_j \bmod l_{m+1}$  is divisible by at least one prime less than or equal to  $p_m$ . Thus the two sets are disjoint. □

**Lemma 9.** *The degree of any vertex in  $E$  is at most five.*

*Proof.* For any  $z \in \mathbb{Z}$  the degree of  $z$  is 2 plus twice the number of  $m$  such that  $z \in L_m$  plus the number of  $m$  such that  $z \in R_m$ . Thus by Lemma 8 the degree of a vertex is at most five. □

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